MATH 5061 Lecture on 3/25/2020

* Important Announcement about Final Exam *

- Final will be a "take-home test" available on May 6, at 2:30 PM on BLACKBOARD
- Submission deadline (by email): May 13, at 2:30 PM.

Remaining Lectures: Apr 1, Apr 8, Apr 15, Apr 22, Apr 29

Last week: $(M^m, \overset{\langle\cdot,\cdot\rangle}{\overset{\parallel}{g}})$ Riemannian manifold

$\rightsquigarrow$ $\exists!$ Levi-Civita connection D on TM s.t. $\begin{cases} Dg \equiv 0 \\ T \equiv 0 \end{cases}$

$\rightsquigarrow$ Riem. curvature $R \in \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(TM))$.

$$R(X,Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z \qquad \forall X,Y,Z \in \mathfrak{X}(M)$$

lower index by g $\Rightarrow$ $R(X,Y,Z,W) := -\langle R(X,Y)Z, W\rangle$ Riem. curvature as $(0,4)$-tensor

Symmetries of R_{ijkl}

(#)
(1) $R_{ijkl} = -R_{jikl} = -R_{ijlk}$

(2) (1^{st} Bianchi) $R_{ijkl} + R_{iklj} + R_{iljk} = 0$

(3) $R_{ijkl} = R_{klij}$

(#) $\Rightarrow$ $\mathcal{R}_p : \Lambda^2 T_pM \times \Lambda^2 T_pM \longrightarrow \mathbb{R}$ curvature operator

$$\mathcal{R}(e_i \wedge e_j, e_k \wedge e_l) := R_{ijkl} \quad \text{for } \{e_i\} \text{ ONB for } T_pM.$$

In particular, $\mathcal{R}(e_i \wedge e_j, e_i \wedge e_j) =: K(\pi)$ where $\pi^2 = \mathrm{span}\{e_i, e_j\} \subseteq T_pM$

symm. (0,2)-tensor

1^{st} trace $\Rightarrow$ Ricci curvature $\mathrm{Ric}(X,Y) := \sum_{i=1}^m R(X, e_i, Y, e_i)$ $[R_{ik} = R_{ijk}{}^{j}]$

2^{nd} trace $\Rightarrow$ Scalar curvature $R := \sum_{i=1}^m \mathrm{Ric}(e_i, e_i) = \sum_{i,j=1}^m R(e_i, e_j, e_i, e_j)$

function

$$[R = R_i{}^i = R_{ij}{}^{ij}]$$

Algebraic decomposition: $\mathrm{Riem} = W + \frac{1}{m-2} \mathring{\mathrm{Ric}} \circ g + \frac{R}{2m(m-1)} g \circ g$

Recall: General setup: (Bianchi identity) $\boxed{D\Omega \equiv 0}$, $D: \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(E)) \to \Gamma(\Lambda^3 T^*M \otimes \mathrm{End}(E))$

locally, this is $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$.

Q: What about in particular for Riem. setting?

Let $p \in M$, and fix a "normal" coord. $x^1, \dots, x^m$ centered at p

ie $x^i(p) = 0$ and "$\Gamma_{ij}^k(0) = 0$" $\Rightarrow$ $\omega(p) = 0$

Write locally: $\Omega^i_j = \frac{1}{2} \sum_{k,\ell} R^i_{jk\ell}\, dx^k \wedge dx^\ell$

$$0 \overset{\text{at } p}{\underset{\text{Bianchi}}{=}} d\Omega^i_j = \frac{1}{2} \sum_{k,\ell,n} \underline{R^i_{jk\ell,n}}\, dx^n \wedge dx^k \wedge dx^\ell \quad \text{at } p$$

$$R^i_{jk\ell,n} = \frac{\partial}{\partial x^n}\left(R^i_{jk\ell}\right) = (D_{\partial_n} R)(\partial_i, \partial_j, \partial_k, \partial_\ell) =: R^i_{jk\ell;n}$$

At p: $R^i_{jk\ell;n} + R^i_{j\ell n;k} + R^i_{jnk;\ell} = 0$

OR lower its index: $(*)$ $\boxed{R_{ijk\ell;n} + R_{ij\ell n;k} + R_{ijnk;\ell} = 0}$ 2nd Bianchi identity

ie $\underbrace{(DR)(X,Y,Z,U,V)}_{= (D_V R)(X,Y,Z,U)} + (DR)(X,Y,U,V,Z) + (DR)(X,Y,V,Z,U) \equiv 0$

Def$^{\underline{n}}$: (Divergence) Let $e_1, \dots, e_m$ be an O.N.B. of T_pM.

- $X \in \mathfrak{X}(M) \Rightarrow \operatorname{div} X(p) := \sum_{i=1}^m \langle D_{e_i} X, e_i \rangle (p)$ $\qquad$ $\operatorname{div} X \in C^\infty(M)$

- $h \in \Gamma(T_2^0 M) \Rightarrow \operatorname{div} h\,(p) := \sum_{i=1}^m (D_{e_i} h)(e_i, \cdot)\,(p)$ $\qquad$ $\operatorname{div} h \in \Omega^1(M)$

  symmetric (0,2)-tensor

Locally $\operatorname{div} X = X^i{}_{;i}$ & $(\operatorname{div} h)_j = h^i{}_{j;i} \overset{\text{O.N.B.}}{=} h_{ij;i}$

Contract i, k in $(*)$ $\Rightarrow$ $(**)$ $\boxed{R_{j\ell;n} + R_{ij\ell n;i} - R_{jn;\ell} = 0}$ contracted 2nd Bianchi

Contract j, l in (**) $\Rightarrow$ $R_{;n} - R_{in;i} - R_{in;i} = 0$

$\Rightarrow$ $2R_{in;i} = R_{;n}$

i.e. (***) $\boxed{2 \operatorname{div} \mathrm{Ric} = dR}$ twice-contracted 2nd Bianchi

$\Updownarrow$

Sometimes, we write: $\boxed{\operatorname{div} G = 0}$ where G is the Einstein tensor

$$G_{ij} := R_{ij} - \frac{R}{2} g_{ij}$$ (0,2)-tensor symm.

(<u>Note</u>: $\mathring{\mathrm{Ric}}_{ij} := R_{ij} - \frac{R}{m} g_{ij}$)

<u>Application of 2nd Bianchi</u>:

<u>Def^{n}</u>: (M^m, g) is Einstein if $\mathrm{Ric}(g) = \frac{R_0}{m} g$ (E) "constant Ricci" for some constant $R_0 \in \mathbb{R}$

<u>Remark</u>: It's an important question for which M^m possess Einstein metric g?

<u>Note</u>: (E) take trace $\Rightarrow$ $R(g) \equiv \frac{R_0}{m} \cdot m \equiv R_0$

So, (M, g) Einstein $\Rightarrow$ $\mathring{\mathrm{Ric}} \equiv 0$ & $R \equiv$ const. $(= R_0)$.

<u>Schur's Thm</u>: $\mathring{\mathrm{Ric}} \equiv 0 \Rightarrow (M, g)$ Einsten <u>for $m \geq 3$</u> (Not true when $m = 2$)

<u>Pf</u>: $\mathring{\mathrm{Ric}} \equiv 0 \Rightarrow \mathrm{Ric}_p \overset{(\dagger)}{\equiv} \frac{R_p}{m} g_p$ <u>here</u>: $R = \operatorname{tr}(\mathrm{Ric}) \in C^\infty(M)$ may not be const.

<u>Claim</u>: $R \equiv$ const. when $m \geq 3$.

(***) $\Rightarrow$ $\frac{1}{2} dR = \operatorname{div} \mathrm{Ric} \overset{(\dagger)}{=} \frac{1}{m} dR$

$m \neq 2 \Rightarrow$ $dR \equiv 0 \Rightarrow R \equiv$ const.

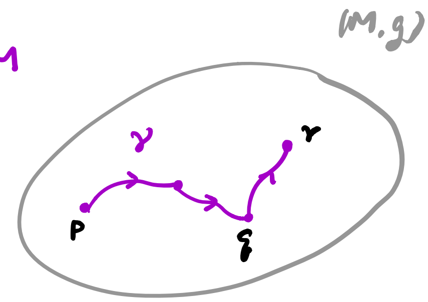
$\square$

<u>Cor</u>: Let $m \geq 3$. Suppose the sectional curvatures are "isotropic", i.e. $\exists f \in C^\infty(M)$

s.t. $K_p(\pi) = f(p) \quad \forall \pi^2 \subseteq T_pM$

Then, $f \equiv$ const. (i.e. (M, g) has const. sectional curvature.)

Metric Geometry on (M^m, g) - connected.



Def$^{\underline{n}}$: The length of a piecewise C^1 curve $\gamma : [0,1] \to M$

is
$$L(\gamma) := \int_0^1 \|\gamma'(t)\|_g \, dt \geq 0$$

where $\|v\|_g := g(v,v)^{1/2}$

We can then define the Riemannian distance function

$$\rho : M \times M \longrightarrow \mathbb{R}_{\geq 0}$$

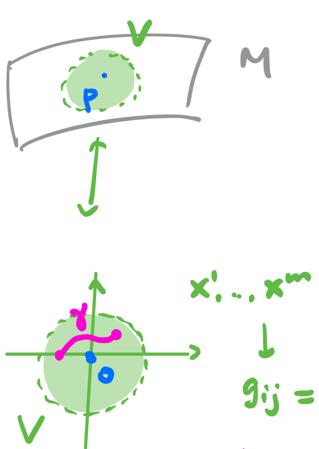
$$\rho(p,q) := \inf \left\{ L(\gamma) \;\middle|\; \begin{array}{l} \gamma : [0,1] \to M \text{ piecewise } C^1 \\ \text{s.t. } \gamma(0) = p, \ \gamma(1) = q \end{array} \right\} \geq 0$$

Fundamental Fact: (M, ρ) is a metric space whose "metric topology" agrees with the "manifold topology" on M.

Reasons: (M,ρ) metric space $\Leftrightarrow$
$$\begin{cases} \rho(p,q) = \rho(q,p) \ \checkmark \\ \rho(p,q) \geq 0 \ \checkmark \text{ with } \text{"="} \text{ only } p = q \\ \rho(p,r) \leq \rho(p,q) + \rho(q,r) \ \checkmark \end{cases}$$

For the "non-trivial" part. Idea: Locally, g metric $\approx$ Eucl. metric

Let $p \in M$, local coord. $x^1, \dots, x^m$ centered at p in a nbd. V of p.



$\downarrow$

$g_{ij} = g(\partial_i, \partial_j)$

δ_{ij} = Eucl. metric

$\exists$ const. $c_1, c_2 > 0$ s.t. $c_1 \delta_{ij} \leq (g_{ij}(x)) \leq c_2 \delta_{ij}$ in V

$\Downarrow$

If $\gamma \subseteq V$, then $c_1 L_\delta(\gamma) \leq L_g(\gamma) \leq c_2 L_\delta(\gamma)$

$\Downarrow$

$$\Rightarrow \underbrace{B_\delta(0, c_1\varepsilon)}_{\text{mfd topo}} \subseteq \underbrace{B_g(0,\varepsilon)}_{\text{metric topology}} \subseteq \underbrace{B_\delta(0, c_2\varepsilon)}_{\text{manifold topo.}} \quad \forall \varepsilon > 0 \text{ small.}$$

$\blacksquare$

Remark: If (M,g) is "nice", then $\rho(p,q)$ is realized by some γ.

Idea: These "straight lines" γ are geodesics on (M,g).

Recall: $\gamma : [0,1] \to (M,g)$ geodesic iff $D_{\gamma'} \gamma' \equiv 0$.

i.e. $$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(x(t)) \frac{dx^i}{dt}\frac{dx^j}{dt} = 0$$ in local coord $\gamma(t) = (x^1(t), \dots, x^m(t))$

Note: $Dg \equiv 0 \Rightarrow \|\gamma'(t)\|_g \equiv \text{const.} = 1$ arc-length parametrization

FACT: Any C^1 regular ($\gamma' \neq 0$) curve can be reparametrized by arc-length.

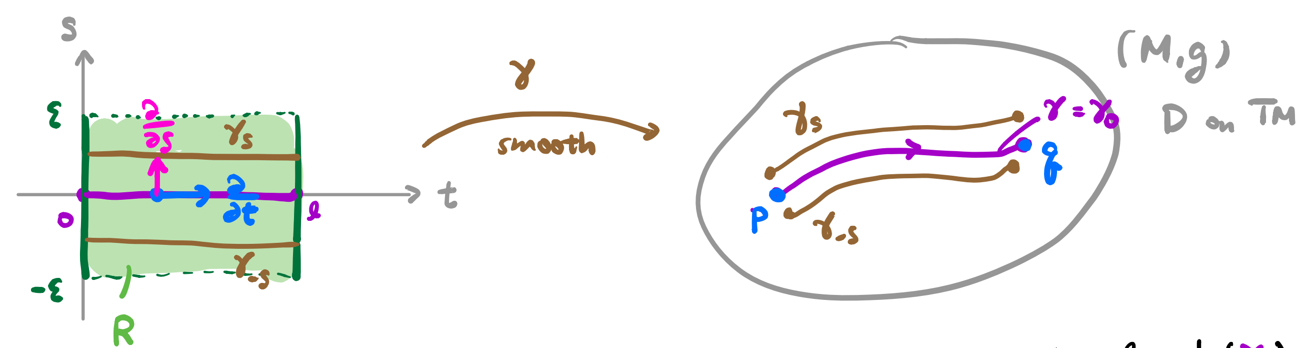
So: "geodesics" = "straight lines" on (M,g) + const. speed parametrization

Prop: Up to reparametrization, a curve $\gamma : [0,\ell] \to (M,g)$ is a geodesic iff γ is a "critical point" to the length functional L (with end pt fixed)

i.e. If $\gamma_s : [0,\ell] \to (M,g)$, $s \in (-\varepsilon, \varepsilon)$, is ANY 1-parameter family of (smooth) curves with $\gamma_0 = \gamma$, then

$$\left.\frac{d}{ds}\right|_{s=0} L(\gamma_s) = 0$$

Picture:



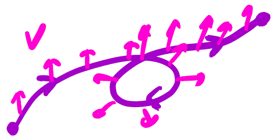
Proof: **WLOG:** Assume $\gamma = \gamma_0(t) : [0,\ell] \to (M,g)$ is p.b.a.l., i.e. $\ell = L(\gamma_0)$.

Consider the 1-parameter family as

$$\gamma(t,s) := \gamma_s(t) : \underbrace{[0,\ell] \times (-\varepsilon,\varepsilon)}_{R} \longrightarrow M$$

Note: (t,s) is a coord. system on R

- γ induces a pullback bundle on $\mathbb{R}$



$$\begin{array}{ccc} \underbrace{g, D}_{\text{pullback metric \& connection}} \quad \gamma^* TM & \longrightarrow & TM, g, D \\ \downarrow & & \downarrow \\ \mathbb{R} & \xrightarrow{\gamma} & M \end{array}$$

FACT: D is also a torsion-free, metric compatible connection on $\gamma^* TM$.

Then, $L(\gamma_s) := \int_0^\ell \|\gamma_s'(t)\|_g \, dt := \int_0^\ell \|\underbrace{\gamma_*(\tfrac{\partial}{\partial t})}_{\text{velocity field}}\|_g \, dt.$

Denote: $V := \gamma_*\left(\frac{\partial}{\partial s}\Big|_{s=0}\right) \in \Gamma(\gamma_0^* TM)$ variation field along γ_0

We compute:

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \int_0^\ell \frac{\langle D_{\frac{\partial}{\partial s}} \gamma_*(\frac{\partial}{\partial t}), \gamma_*(\frac{\partial}{\partial t}) \rangle}{\underbrace{\|\gamma_*(\frac{\partial}{\partial t})\|}_{\equiv 1 \;\because\; \gamma_0 \text{ p.b.a.l.}}}\Bigg|_{s=0} dt$$

$\because$ D is torsion-free
$\because$ $[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] \equiv 0$

$$= \int_0^\ell \langle D_{\frac{\partial}{\partial t}} \gamma_*(\tfrac{\partial}{\partial s}), \gamma_*(\tfrac{\partial}{\partial t}) \rangle \Big|_{s=0} dt$$

$$= \int_0^\ell \langle D_{\frac{\partial}{\partial t}} V(t), \gamma_0'(t) \rangle \, dt$$

$\because$ D is metric compatible

$$= \int_0^\ell \frac{\partial}{\partial t}\left(\langle V(t), \gamma_0'(t) \rangle\right) - \langle V(t), D_{\frac{\partial}{\partial t}} \gamma_0'(t) \rangle \, dt$$

$$= \langle V(t), \gamma_0'(t) \rangle \Big|_{t=0}^{t=\ell} - \int_0^\ell \langle V, D_{\frac{\partial}{\partial t}} \gamma_0' \rangle (t) \, dt$$

$$\boxed{\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \langle V(t), \gamma_0'(t) \rangle \Big|_{t=0}^{t=\ell} - \int_0^\ell \langle V, D_{\frac{\partial}{\partial t}} \gamma_0' \rangle (t) \, dt}$$

1st variation formula for length functional

If end points are fixed, then $V(0) = 0 = V(\ell)$ and

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = 0 \iff -\int_0^\ell \langle V, D_{\frac{\partial}{\partial t}} \gamma_0' \rangle \, dt = 0 \iff D_{\gamma_0'} \gamma_0' \equiv 0$$

$\forall$ variation $\{\gamma_s\}$ of γ_0 $\qquad$ $\forall\, V \in \Gamma(\gamma_0^* TM)$ $\qquad$ geodesic!

Exponential Map & Geodesic Normal Coordinates

Recall: Fix $p \in M$, $v \in T_pM$, then

$$(\text{IVP}): \begin{cases} D_{\gamma'}\gamma' \equiv 0 \text{ along } \gamma \\ \gamma(0) = p \\ \gamma'(0) = v \end{cases}$$

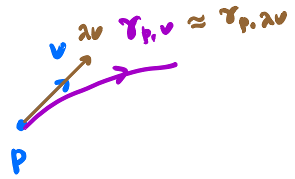
has unique sol^{n} on some short interval:

$$\gamma_{p,v} := \gamma : (-\varepsilon, \varepsilon) \longrightarrow M$$

Note: ε depends on p, v.

BuT, $\gamma_{p,v}$ and ε depends smoothly on p, v.

Homogenity of geodesics: $\gamma_{p,\lambda v}(t) = \gamma_{p,v}(\lambda t) \quad \forall \lambda \in \mathbb{R}$



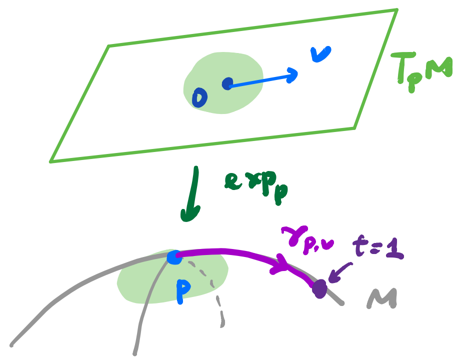
$$\gamma_{p,v} : (-\varepsilon, \varepsilon) \to M$$

$$\Rightarrow \gamma_{p,\lambda v} : \left(-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda}\right) \to M$$

By cptness, the exponential map of (M, g) at $p \in M$:

$$\begin{array}{cccc} \exp_p : & \underset{0}{\mathcal{U}} \subseteq T_pM & \longrightarrow & \underset{p}{M} \\ & v & \longmapsto & \gamma_{p,v}(1) \end{array}$$

is well-defined in some nbd. $\mathcal{U} \subseteq T_pM$ (and smooth)



Prop: $d(\exp_p)\big|_0 : T_pM \to T_pM$ is the identity map.

Pf: $$d(\exp_p)_0(v) := \frac{d}{ds}\Big|_{s=0} \exp_p(sv) := \frac{d}{ds}\Big|_{s=0} \gamma_{p,sv}(1)$$

$$= \frac{d}{ds}\Big|_{s=0} \gamma_{p,v}(s) = \gamma'_{p,v}(0) = v \qquad \blacksquare$$

By I.F.T., $\exp_p$ is a local diffeo at 0.

$$\exp_p : \mathcal{U} \xrightarrow[\cong]{\text{diffeo.}} V$$

defines the geodesic normal coordinates near p.